Dynamics of the Bose-Einstein condensation: Analogy with the collapse dynamics of a classical self-gravitating Brownian gas

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We consider the dynamics of a gas of free bosons within a semiclassical Fokker-Planck equation for which we give a physical justification. In this context, we find a striking similarity between the Bose-Einstein condensation in the canonical ensemble, and the gravitational collapse of a gas of classical self-gravitating Brownian particles. The paper is mainly devoted to the complete study of the Bose-Einstein "collapse" within this model. We find that at the Bose-Einstein condensation temperature T_c, the chemical potential $\mu(t)$ vanishes exponentially with a universal rate that we compute exactly. Below T_c, we show analytically that $\sqrt{\mu(t)}$ vanishes linearly in a finite time t_{coll}. After t_{coll}, the mass of the condensate grows linearly with time and saturates exponentially to its equilibrium value for large time. We also give analytical results for the density scaling functions, for the corrections to scaling, and for the exponential relaxation time. Finally, we find that the equilibration time (above T_c) and the collapse time t_{coll} (below T_c) both behave like $-T_c^{-3}\ln|T-T_c|$, near T_c.

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I. INTRODUCTION

The Bose-Einstein condensation is a fundamental result of quantum statistics [1]. Below a critical temperature T_c, a finite fraction of bosons enters the ground state and a Bose-Einstein condensate (BEC) forms [2]. Observation of Bose-Einstein condensation in cold atomic samples was first reported in 1995 by three different groups [3–5] in a vapor of spin-polarized ^{87}Rb, ^{7}Li, and ^{23}Na atoms. Apart from laboratory experiments, another interesting application of Bose condensation is related to the problem of boson star formation from the dark matter in the Universe [6]. In that case, Bose star formation involves the axion as a dark matter particle candidate [7].

Since its discovery [8,9], several authors have attempted to develop kinetic models to describe the dynamical process of the Bose-Einstein condensation. The dynamical evolution of a homogeneous gas of bosons interacting only via "collisions" can be studied using an appropriate form of the Boltzmann equation [10–12] which takes into account the specificities of the Bose statistics. Since the gas is isolated and the energy conserved, this model describes a *microcanonical* situation. For this model, the Bose-Einstein condensation in momentum space has been considered by Semikoz and Tkachev [13] and Lacaze *et al.* [14]. Alternatively, the *canonical* evolution of a system of noninteracting bosons coupled to a thermostat imposing the temperature can be described by a semiclassical Fokker-Planck equation [15–17]. For $T>T_c$, where T_c is the critical temperature, this equation converges towards the Bose-Einstein distribution. The main purpose of this paper is a complete study of this equation below the critical temperature T_c, in order to understand the dynamics of the formation of the condensate. Our study is restricted to a spatially homogeneous system without interaction at arbitrary temperature. Alternatively, the collapsing dynamics of a trapped Bose-Einstein condensate (BEC) with attractive interaction is often analyzed in terms of the Gross-Pitaevskii (GP) or nonlinear Schrödinger equation (see, for instance [18]). This describes the formation of a spatially inhomogeneous condensate at $T=0$. The GP equation can display a self-similar collapse, but it occurs in position space while the system that we shall consider is spatially homogeneous and the condensation occurs in **k**-space.

We thus consider a population of free bosons in d dimensions with dispersion relation $\varepsilon(\mathbf{k})=\frac{k^2}{2m}$ (we set $m=1$ in the following). This system is assumed to be strongly coupled to a thermal bath at temperature T. Initially, at $t=0$, the system is prepared in an initial state which can be, for instance, the equilibrium distribution at some high temperature $T_0 \gg T$. At infinite time, we expect the system to reach thermal equilibrium at temperature T, characterized by the Bose-Einstein distribution if $T>T_c$ (where T_c is the Bose-Einstein condensation temperature)

$$\rho_{BE}(\mathbf{k})=\frac{1}{\exp\left(\frac{\beta k^2}{2}+\mu\right)-1}, \tag{1}$$

where μ is the chemical potential. Below T_c, the equilibrium distribution is the Bose-Einstein distribution with $\mu=0$ plus a Dirac peak at $\mathbf{k}=\mathbf{0}$ (the condensate) containing the rest of the mass.

A. Bosonic Fokker-Planck equation

In this paper, we are interested in the temporal evolution of the occupation number at momentum $\mathbf{k}$, that we call $\rho(\mathbf{k},t)$, from an arbitrary initial state to the final equilibrium state described above. If our particles were classical instead of bosons (of course no condensate can appear in that case), the coupling to the thermal bath can be modelized by a random force and a friction, and the evolution equation for the momentum of a given particle is described by the Langevin dynamics

$$\frac{d\mathbf{k}}{dt} = -\frac{\mathbf{k}}{\xi} + \mathbf{f}(t), \tag{2}$$

where $\mathbf{f}(t)$ is a delta-correlated random force, whose components satisfy

$$\langle f_i(t) f_j(t')\rangle = 2D\delta_{ij}\delta(t-t'), \quad i,j \in [1, \ldots, d]. \tag{3}$$

In order to recover the equilibrium equipartition theorem, we must impose the Einstein relation

$$D = \frac{T}{\xi}, \tag{4}$$

where we have set the Boltzmann constant $k_B = 1$. The Fokker-Planck equation describing the temporal evolution of the momentum distribution reads

$$\frac{\partial \rho}{\partial t} = \frac{1}{\xi}\nabla_{\mathbf{k}}[T\nabla_{\mathbf{k}}\rho + \rho \mathbf{k}]. \tag{5}$$

Ultimately, the momentum distribution converges to the expected Boltzmann distribution

$$\rho_B(\mathbf{k}) = Z^{-1} \exp\left(-\frac{\beta k^2}{2}\right). \tag{6}$$

Now, coming back to the Bose gas, we ask whether a stochastic dynamics can be introduced, which accurately describes the actual evolution [48]. Restraining ourselves to noninteracting bosons, we shall see that one can give a reasonable answer to this question. This will permit the description of the dynamical formation of the condensate at and below T_c, which is the main purpose of this paper.

Several authors have considered this problem [16,17]. Let us introduce a stochastic dynamics by defining the rate at which particles with momentum $\mathbf{k}$ get a new momentum $\mathbf{k}'$. Following Kaniadakis *et al.* [16], we assume the following form of this rate:

$$W(\mathbf{k} \to \mathbf{k}') = w(\mathbf{k}, \mathbf{k}-\mathbf{k}')a[\rho(\mathbf{k},t)]b[\rho(\mathbf{k}',t)]. \tag{7}$$

Contrary to classical dynamics, we assume a dependence of $W(\mathbf{k}\to\mathbf{k}')$ with $\rho(\mathbf{k},t)$ and $\rho(\mathbf{k}',t)$. Otherwise, we would simply recover the Fokker-Planck equation Eq. (5), as will be shown below. In fact, $W(\mathbf{k}\to\mathbf{k}')$ being a *rate* of departure from the state of momentum $\mathbf{k}$, it seems reasonable to assume that it should not depend on the population of this initial state. Hence we set $a(\rho)=1$. Then, one can write the master equation describing the time evolution of the particle distribution as

$$\frac{\partial \rho}{\partial t} = \int [\rho(\mathbf{k}',t)W(\mathbf{k}' \to \mathbf{k}) - \rho(\mathbf{k},t)W(\mathbf{k} \to \mathbf{k}')]d\mathbf{k}'. \tag{8}$$

Assuming that the evolution is sufficiently slow and local, such that the dynamics only permits values of $\mathbf{k}'$ close to $\mathbf{k}$, one can develop $W(\mathbf{k}\to\mathbf{k}')$ in powers of $\mathbf{k}-\mathbf{k}'$ in Eq. (8). After some algebra, and proceeding along the line of [16], one obtains a Fokker-Planck-like equation

$$\frac{\partial \rho}{\partial t} = \nabla_{\mathbf{k}}[b(\rho)\nabla_{\mathbf{k}}(D\rho) + \rho b(\rho)\mathbf{J} - D\rho b'(\rho)\nabla_{\mathbf{k}}\rho], \tag{9}$$

where assuming isotropy, we obtain

$$D(\mathbf{k}) = \frac{1}{2d}\int w(\mathbf{k},\Delta\mathbf{k})(\Delta\mathbf{k})^2 d(\Delta\mathbf{k}), \tag{10}$$

$$\mathbf{J}(\mathbf{k}) = -\int w(\mathbf{k},\Delta\mathbf{k})\Delta\mathbf{k}\, d(\Delta\mathbf{k}). \tag{11}$$

We now make the simplification that the diffusion coefficient $D(\mathbf{k})$ is momentum-independent, and that the current $\mathbf{J}$ is simply proportional to the particle's velocity or momentum $\mathbf{k}$:

$$D(\mathbf{k}) = \frac{T}{\xi}, \tag{12}$$

$$\mathbf{J}(\mathbf{k}) = \frac{\mathbf{k}}{\xi}. \tag{13}$$

Going from Eq. (10) to Eq.(12) is a standard assumption in kinetic theory. It is exactly achieved when $w(\mathbf{k},\Delta\mathbf{k})$ is a symmetric analytic function of $(\Delta\mathbf{k}+\mathbf{k}\Delta t/\xi)/\sqrt{\Delta t}$, where Δt is the discretized time step (ultimately $\Delta t \to 0$). This function must also decay fast enough so that its two first moments are well-defined. In the case of the usual Brownian motion based on the Langevin equations (2), this function is simply a Gaussian. Using Eqs. (12) and (13), Eq. (9) becomes

$$\frac{\partial \rho}{\partial t} = \frac{1}{\xi}\nabla_{\mathbf{k}}[T(b(\rho)-\rho b'(\rho))\nabla_{\mathbf{k}}\rho + \rho b(\rho)\mathbf{k}]. \tag{14}$$

Note that choosing $b(\rho)=1$, i.e., a transition rate which does not depend on the population of the new momentum state, leads to the standard Fokker-Planck equation, Eq. (5), for classical particles. In the general case, the stationary distribution for $\mathbf{k}\neq\mathbf{0}$ satisfies

$$\left(\frac{1}{\rho} - \frac{b'(\rho)}{b(\rho)}\right)\nabla_{\mathbf{k}}\rho = -\beta\mathbf{k}, \tag{15}$$

which can be integrated, leading to

$$\frac{\rho}{b(\rho)} = \exp\left(-\frac{\beta k^2}{2} - \mu\right), \tag{16}$$

where μ is an integration constant. In order to recover the Bose-Einstein distribution Eq. (1), we must make the unique choice

$$b(\rho) = 1 + \rho. \tag{17}$$

We conclude that the classical rate is amplified by the factor $b(\rho) > 1$, which translates the tendency of bosons to fill already occupied states. Note that the choice $b(\rho)=1-\rho$ leads to the Fermi-Dirac distribution [17], where $b(\rho)$ now suppresses the probability to hop to an already occupied state, as can be expected for fermions (note that a generalized Fokker-Planck equation incorporating an exclusion principle has been introduced independently in [27] in the very different context of the violent relaxation of collisionless stellar systems and 2D vortices). This approach has been extended to the case of intermediate statistics, associated to $b(\rho)=1+\kappa\rho$ ($\kappa\in[-1,1]$) [17]. It can also be generalized in order to recover Tsallis statistical thermodynamics at equilibrium from a general kinetic equation [17,28].

Focusing on the bosonic case, we finally obtain the associated Fokker-Planck equation

$$\frac{\partial\rho}{\partial t}=\frac{1}{\xi}\nabla_{\mathbf{k}}[T\nabla_{\mathbf{k}}\rho+\rho(1+\rho)\mathbf{k}]. \tag{18}$$

This bosonic Kramers equation could be directly obtained from a modified Langevin equation

$$\frac{d\mathbf{k}}{dt}=-\frac{\mathbf{k}}{\xi}[1+\rho(\mathbf{k},t)]+\mathbf{f}(t), \tag{19}$$

where the friction term which tends to move the momentum toward $\mathbf{k}=\mathbf{0}$ now increases with the occupation number at momentum $\mathbf{k}$. In this context, the origin of the Bose-Einstein instability is quite clear.

Finally, for later reference, we write the bosonic Fokker-Planck equation for a spherically symmetric solution. Setting $\xi=1$, we get

$$\frac{\partial\rho}{\partial t}=T\left(\frac{\partial^2\rho}{\partial k^2}+\frac{d-1}{k}\frac{\partial\rho}{\partial k}\right)+d\rho(1+\rho)+k(2\rho+1)\frac{\partial\rho}{\partial k}. \tag{20}$$

For the integrated density

$$M(k,t)=\int_0^k\rho(k',t)k'^{d-1}dk', \tag{21}$$

we obtain

$$\frac{\partial M}{\partial t}=T\left(\frac{\partial^2 M}{\partial k^2}-\frac{d-1}{k}\frac{\partial M}{\partial k}\right)+k\frac{\partial M}{\partial k}\left(\frac{1}{k^{d-1}}\frac{\partial M}{\partial k}+1\right). \tag{22}$$

It has to be noted that the bosonic Fokker-Planck equation (18) valid for noninteracting bosons in contact with a heat bath (canonical ensemble) is the counterpart of the bosonic Boltzmann equation [10–12] valid for an isolated system of interacting bosons (microcanonical ensemble). These kinetic equations are *semiclassical* equations where quantum mechanics enters only through the Bose-Einstein statistics. The bosonic Boltzmann equation can be derived from a fully quantum treatment as discussed in Sec. VI. The justification of the bosonic Fokker-Planck equation from a fully quantum mechanics treatment would be interesting but is beyond the scope of this paper.

B. Analogy with a self-gravitating gas of Brownian particles

In a series of recent papers [28–32], two of the present authors have introduced and systematically studied the dynamical properties of a self-gravitating gas of Brownian particles in all spatial dimensions. This is the *canonical* version of the original and certainly more challenging problem of self-gravitating Newtonian particles in the *microcanonical* ensemble. In the overdamped limit, the Langevin equation for the position in real space of a particle reads

$$\frac{d\mathbf{r}}{dt}=-\frac{\nabla\Phi}{\xi}+\mathbf{f}(t), \tag{23}$$

where the gravitational potential $\Phi(\mathbf{r},t)$ must be computed self-consistently using the Poisson equation

$$\Delta\Phi(\mathbf{r},t)=S_d G\rho(\mathbf{r},t), \tag{24}$$

where S_d is the surface of the unit d-dimensional sphere, and G is Newton's constant. The associated Fokker-Planck-Poisson (or Smoluchowski-Poisson) system is obtained straightforwardly

$$\frac{\partial\rho}{\partial t}=\frac{1}{\xi}\nabla[T\nabla\rho+\rho\nabla\Phi], \tag{25}$$

$$\Delta\Phi=S_d G\rho. \tag{26}$$

As discussed in [33], these equations also describe the chemotaxis of bacterial populations in biology, by a proper reinterpretation of the parameters. From now on, we get rid of nonessential constants by setting

$$G=\xi=S_d=1. \tag{27}$$

For a time-dependent radial solution, the system of equations, Eqs. (25) and (26), can be put into a unique equation [30]

$$\frac{\partial\rho}{\partial t}=T\left(\frac{\partial^2\rho}{\partial r^2}+\frac{d-1}{r}\frac{\partial\rho}{\partial r}\right)+\rho^2+\frac{M}{r^{d-1}}\frac{\partial\rho}{\partial r}, \tag{28}$$

where

$$M(r,t)=\int_0^r\rho(r')r'^{d-1}dr' \tag{29}$$

is the integrated density. Actually, the equation for $M(r,t)$ looks even simpler

$$\frac{\partial M}{\partial t}=T\left(\frac{\partial^2 M}{\partial r^2}-\frac{d-1}{r}\frac{\partial M}{\partial r}\right)+\frac{M}{r^{d-1}}\frac{\partial M}{\partial r}. \tag{30}$$

It is clear that Eqs. (28) and (30) are strikingly similar to the dynamical equations found in the context of the Bose-Einstein condensation Eqs. (20) and (22). Apart from the identical diffusion term, we note that the nonlinear terms have the same dimension as ρ^2 in Eqs. (20) and (28) and dimension $M^2\times(k,r)^{-d}$ in Eqs. (22) and (30) (we shall see later that the +1 term at the end of Eq. (22) is essentially irrelevant as far as the dynamics of the Bose-Einstein condensation at or below T_c is concerned).

Before summarizing our results concerning the dynamics of a gas of bosons in $d=3$ described by Eqs. (20) and (22), we would like to mention only the main results obtained for a self-gravitating Brownian gas obeying Eqs. (28) and (30). The comparison with this surprisingly close model will certainly prove interesting.

(1) In $d\geq 3$, Eqs. (28) and (30) reproduce the known equilibrium profile for $T\geq T_c$. For $T<T_c$, a dynamical instability

arises coinciding with the absence of minimum of the free energy (thermodynamical instability) [29,30].

(2) In $d \geq 3$, for $T < T_c$, the density develops a scaling profile $\rho(r,t) = \rho_0 f(r/r_0)$, with $\rho_0(t) = T r_0^{-2}(t) \sim (t_{coll} - t)^{-1}$ and $f(x) \sim x^{-2}$ for $x \to +\infty$. Hence, the central density diverges in a finite time t_{coll}. Note that f can be calculated analytically in all dimensions, and that corrections to scaling have been evaluated [29,30].

(3) In $d \geq 3$, for $T < T_c$, and for $t \geq t_{coll}$, the central density is swallowed by an emerging condensate at $\mathbf{r} = \mathbf{0}$, which grows like $M_0(t) \sim (t - t_{coll})^{d/2-1}$. In this postcollapse regime, the residual density obeys a backward dynamical scaling, $\rho(r,t) = \rho_0 g(r/r_0)$, with $\rho_0(t) = T r_0^{-2}(t) \sim (t - t_{coll})^{-1}$ and $g(x) \sim x^{-2}$ for $x \to +\infty$. For large time, the central condensate saturates exponentially to the *total* mass, whereas the residual density vanishes exponentially with the same rate, which has been calculated by semiclassical techniques [31]. We found that the relaxation time τ above T_c diverges like $\tau \sim K_+ (T - T_c)^{-1/2}$, where K_+ is known exactly. Approaching T_c from below, the collapse time t_{coll} diverges like $t_{coll} \sim K_- (T_c - T)^{-1/2}$, where again K_- has been computed analytically [32].

(4) In $d = 2$, at T_c, the density obeys a dynamical scaling with a known scaling function f. The central density diverges like $\rho_0(t) \sim c_1 \exp(c_2 \sqrt{t})$, where c_1 and c_2 are known universal constants [30,34] (in an unbounded domain, the divergence is logarithmic instead of exponential [35]). Below T_c, the collapse dynamics occurs after a finite time t_{coll}. At t_{coll}, a fraction T/T_c of the total mass has condensed at $\mathbf{r} = \mathbf{0}$. Using the Virial theorem, we show that all the mass must collapse at $\mathbf{r} = \mathbf{0}$ in the postcollapse regime [35]. We mention these results in $d = 2$ because the dynamics of the Bose-Einstein condensation in $d = 3$ will share some qualitative analogy with this case.

We thus expect a surprising parallel between the collapse dynamics of a classical self-gravitating Brownian gas, which occurs when the kinetic pressure $P \sim T$ is not strong enough to counterbalance the gravitational attraction, and a bosonic gas, which forms a condensate at $\mathbf{k} = \mathbf{0}$, when the usual Bose-Einstein distribution is not able to accommodate for all the mass, even at zero chemical potential (see Sec. II A). However, we note one fundamental difference between these two systems. For $t \to +\infty$, strictly below T_c all the mass collapses at $\mathbf{r} = \mathbf{0}$ in the gravitational case, whereas the mass of the condensate is temperature-dependent in the case of the Bose-Einstein transition.

C. Summary of results

In Secs. II A and II B, we briefly review some basic results concerning the static properties of the Bose-Einstein condensation, and introduce a simpler model (SM), for which the static and dynamical properties can be analytically studied, and which will be equivalent in the dense region to the original Bose-Einstein model (BEM) described by Eqs. (20) and (22). In Sec. II C, we show that Eqs. (20) and (22) maximize the rate of dissipation of bosonic free energy, and that the dynamical instability exactly coincides with the thermodynamical instability giving rise to the Bose-Einstein condensation below T_c.

In Sec. III, we address the collapse dynamics at and below T_c. In Sec. III A, we compute the density scaling profile which is found to be independent of the temperature $T \leq T_c$, up to a temperature-dependent multiplicative factor. In Sec. III B, we show that at $T = T_c$ the central density (at $\mathbf{k} = \mathbf{0}$) diverges exponentially with time, with a rate which can be calculated analytically. We compute the exponentially decaying corrections to the final Bose-Einstein distribution with $\mu = 0$. In Sec. III C, we show that for $T < T_c$, the chemical potential vanishes in a finite time and that $\sqrt{2T\mu(t)} \sim c(T)(t_{coll} - t)$, where $c(T)$ is computed analytically close to T_c. We show that t_{coll} diverges logarithmically as T approaches T_c from below. In Sec. III D, we address the specific collapse dynamics obtained strictly at $T = 0$.

In Sec. IV, we study the postcollapse dynamics arising for $T < T_c$, and for times $t > t_{coll}$. We show that the condensate mass initially grows linearly, with a slope which can be calculated exactly near T_c. For large times, the mass of the condensate saturates exponentially fast to its equilibrium value, with a rate analytically known close to T_c.

In Sec. V, we consider the relaxation time above T_c, which is computed in different limits. This relaxation time does not diverge at T_c, but the time after which this relaxation regime occurs (that we call the equilibration time) diverges logarithmically as the temperature approaches T_c from above.

II. EQUILIBRIUM PROPERTIES AND A SIMPLER MODEL IN A BOX

A. Bose-Einstein condensation for a gas of free bosons

Let us briefly repeat the few steps leading to the Bose-Einstein condensation. Considering free bosons, the equilibrium occupation number at momentum $\mathbf{k}$ is

$$\rho_{BE}(\mathbf{k}) = \frac{1}{\exp\left(\frac{\beta k^2}{2} + \mu\right) - 1}. \tag{31}$$

The total mass is fixed to unity, and the chemical potential μ is defined implicitly by the total mass constraint (for simplicity, we set the geometrical factor $S_d = 1$, which amounts to fixing $M = S_d = 1$)

$$M = \int_0^\infty \rho_{BE}(k) k^{d-1} dk = 1, \tag{32}$$

which can be rewritten as the identity

$$\beta^{d/2} = \int_0^\infty \frac{k^{d-1}}{\exp\left(\frac{k^2}{2} + \mu\right) - 1} dk. \tag{33}$$

Below a certain temperature T_c, this distribution cannot include all the mass. The critical temperature corresponds to $\mu = 0$. Specializing to the case of dimension $d = 3$ which is the focus of this paper (there is no condensation for $d < 3$), we find

$$\beta_c = \left(\frac{\pi}{2}\right)^{1/3} \zeta^{2/3}\left(\frac{3}{2}\right), \quad \beta_c \approx 2.20494\ldots, \tag{34}$$

where $\zeta(x)=\sum_{k=1}^{+\infty} k^{-x}$ is the Riemann ζ-function. Below T_c, the momentum distribution is the Bose-Einstein distribution with $\mu=0$ plus a Dirac peak at $\mathbf{k=0}$ containing the excess of mass M_0, with

$$M_0 = 1 - \left(\frac{\beta_c}{\beta}\right)^{3/2} \sim \frac{3}{2}\frac{T_c - T}{T_c}, \quad \frac{T-T_c}{T_c} \ll 1. \tag{35}$$

Above T_c, we find the following asymptotics which will be useful later:

$$\mu \sim \frac{9}{8\pi^2} T_c^3 \left(\frac{T-T_c}{T_c}\right)^2, \quad \frac{T-T_c}{T_c} \ll 1, \tag{36}$$

and

$$\mu \sim \frac{3}{2}\ln T, \quad T \to +\infty. \tag{37}$$

B. A simpler model in a box

We now introduce a simpler model (hereafter denoted SM), which will share the same properties as our original model, as far as the condensation dynamics properties are concerned. In the kinetic equation Eq. (18), we replace $(\rho+1)$ with ρ, which should be valid in the dense region around $\mathbf{k=0}$, during the formation of the condensate. We obtain

$$\frac{\partial \rho}{\partial t} = \nabla_{\mathbf{k}}[T\nabla_{\mathbf{k}}\rho + \rho^2 \mathbf{k}], \tag{38}$$

and the following equation for a radial solution

$$\frac{\partial \rho}{\partial t} = T\left(\frac{\partial^2 \rho}{\partial k^2} + \frac{d-1}{k}\frac{\partial \rho}{\partial k}\right) + d\rho^2 + 2k\rho\frac{\partial \rho}{\partial k}. \tag{39}$$

The integrated density satisfies

$$\frac{\partial M}{\partial t} = T\left(\frac{\partial^2 M}{\partial k^2} - \frac{d-1}{k}\frac{\partial M}{\partial k}\right) + \frac{1}{k^{d-2}}\left(\frac{\partial M}{\partial k}\right)^2, \tag{40}$$

which is even more similar to the dynamical equation Eq. (30) for a gas of self-gravitating Brownian particles than Eq. (22). The stationary solution can be easily calculated

$$\rho(\mathbf{k}) = \frac{1}{\frac{\beta k^2}{2} + \mu}, \tag{41}$$

which amounts to replacing the function $\exp x$ in the Bose-Einstein distribution by the function $(1+x)$. Note that the distribution (41) is the Lorentzian. Since this distribution $\rho(\mathbf{k})$ is not integrable up to infinity in $d \geq 2$, we introduce a momentum cutoff

$$k \leq \Lambda. \tag{42}$$

As before, the total mass constraint reads

$$M = \int_0^{\Lambda} \frac{k^{d-1}}{\frac{\beta k^2}{2} + \mu} dk = 1. \tag{43}$$

From now on, we set $d=3$, as well as in the rest of the paper. Then, the integrated density can be explicitly calculated

$$M(k) = 2T\left[k - \sqrt{2\mu T}\arctan\left(\frac{k}{\sqrt{2\mu T}}\right)\right]. \tag{44}$$

The density profile Eq. (41) can accommodate for all the mass up to the temperature

$$\beta_c = 2\Lambda. \tag{45}$$

Below T_c, a Dirac peak appears at $\mathbf{k=0}$, containing the excess mass

$$M_0 = \frac{T_c - T}{T_c}. \tag{46}$$

Just above T_c, the chemical potential vanishes in a manner similar to the original Bose-Einstein model (BEM),

$$\mu \sim \frac{1}{2\pi^2} T_c^3 \left(\frac{T-T_c}{T_c}\right)^2, \quad \frac{T-T_c}{T_c} \ll 1, \tag{47}$$

whereas at high temperatures, the two models are qualitatively different, as the particle's momentum cannot spread up to infinity in the SM. Hence the chemical potential converges at high temperatures

$$\mu \to \frac{\Lambda^3}{3}, \quad T \to +\infty. \tag{48}$$

C. Some properties of the bosonic Fokker-Planck equation

Equation (18) belongs to a general class of nonlinear Fokker-Planck (NFP) equations considered by Kaniadakis [17], Frank [36], and Chavanis [37]. These nonlinear Fokker-Planck equations arise when the coefficients of diffusion, mobility, and friction depend explicitly on the distribution function. There are different ways to write these NFP equations. One convenient form for our present purposes is [37]

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}_* = \nabla\left[\frac{1}{\xi}\left(T\nabla\rho + \frac{1}{C''(\rho)}\nabla\Phi\right)\right], \tag{49}$$

where $C(\rho)$ is a convex function and $\xi>0$ can depend on $\mathbf{r}$ and t, e.g., on $\rho(\mathbf{r},t)$. Here, $\Phi(\mathbf{r})$ is a fixed external potential but we can also consider the situation where $\Phi(\mathbf{r},t)$ is generated by $\rho(\mathbf{r},t)$ [37], like in the case of self-gravitating Brownian particles (see Sec. I B). Taking $C(\rho)=\rho\ln\rho - \frac{1}{\kappa}(1+\kappa\rho)\ln(1+\kappa\rho)$, we recover the bosonic Fokker-Planck equation (18) for $\kappa=+1$, the fermionic Fokker-Planck equation for $\kappa=-1$, and the classical Fokker-Planck equation for $\kappa=0$ (for other values of κ, this describes intermediate "quon" statistics). For the simpler model Eq. (38), we have $C(\rho)=-\ln\rho$ describing logotropes [38]. Introducing a generalized free energy

$$F = E - TS = \int \rho\Phi d\mathbf{r} + T\int C(\rho) d\mathbf{r}, \quad (50)$$

one can show that $F(t)$ is monotonically decreasing: $\dot{F} = -\int \xi C''(\rho)\mathbf{J}_*^2 d\mathbf{r} \le 0$. Therefore, if F is bounded from below, the density $\rho(\mathbf{r},t)$ will converge, for $t \to +\infty$, to the stationary solution $\rho_{eq}(\mathbf{r})$ such that $\mathbf{J}_* = \mathbf{0}$ (in that case F is called a Lyapunov functional). The stationary solution of Eq. (49) is determined by

$$\rho_{eq}(\mathbf{r}) = (C')^{-1}[-\beta\Phi(\mathbf{r}) - \alpha], \quad (51)$$

where $\beta = 1/T$ and α is an integration constant. This stationary solution is a critical point of free energy Eq. (50) at fixed mass, i.e., it satisfies $\delta F + \alpha T \delta M = 0$, where αT is a Lagrange multiplier. Moreover, it is shown in [37] that a stationary solution of Eq. (49) is linearly dynamically stable if, and only if, it is a *minimum* of F at fixed mass. In fact, when $\Phi(\mathbf{r})$ is an external potential, the critical points of F at fixed mass are necessarily minima since $\delta^2 F = \frac{1}{2}T\int C''(\rho)(\delta\rho)^2 d\mathbf{r} \ge 0$. Therefore, if a critical point of free energy exists it is the only minimum of F and, consequently, F is bounded from below: $F[\rho] \ge F[\rho_{eq}]$. In that case, the dynamical equation, Eq. (49), will relax towards $\rho_{eq}(\mathbf{r})$ for $t \to +\infty$.

The NFP equation, Eq. (49), can be justified in several different ways [37]. It can be obtained from the linear thermodynamics of Onsager by relating the current to the gradient of a "chemical potential" $\alpha(\mathbf{r},t) = -\beta\Phi(\mathbf{r}) - C'(\rho)$ that is uniform at equilibrium [see Eq. (51)], writing $\mathbf{J}_* = 1/[\beta\xi C''(\rho)]\nabla\alpha$. The current can also be expressed as the functional derivative of the free energy, writing $\mathbf{J}_* = -1/[\xi C''(\rho)]\nabla(\delta F/\delta\rho)$. Alternatively, the NFP equation can be obtained by writing $\partial_t \rho = -\nabla \cdot \mathbf{J}$ and looking for the optimal current $\mathbf{J}_*$ which maximizes the rate of free energy dissipation $\dot{F}[\mathbf{J}] = \int [TC''(\rho)\nabla\rho + \nabla\Phi]\mathbf{J} d\mathbf{r}$ for a bounded function $E_d[\mathbf{J}] = \frac{1}{2}\int \xi C''(\rho)\mathbf{J}^2 d\mathbf{r}$ preventing arbitrarily large values of the current $\mathbf{J}$. This is the variational version of the linear thermodynamics of Onsager. The optimal current, that is solution of $\delta\dot{F} + \delta E_d = 0$, is the one appearing in Eq. (49) and it satisfies $\dot{F}[\mathbf{J}_*] = -2E_d[\mathbf{J}_*] \le 0$. It indeed leads to the most negative value of $\dot{F}$ (under constraints) since $\delta^2(\dot{F} + E_d) = \delta^2 E_d = \frac{1}{2}\int \xi C''(\rho)(\delta\mathbf{J})^2 d\mathbf{r} \ge 0$. These methods emphasize the (generalized) thermodynamical structure of the NFP equation.

III. COLLAPSE DYNAMICS

A. General scaling solutions for $T \le T_c$

We now consider the collapse dynamics occurring when the system is suddenly quenched to T_c or below T_c from high temperature. It is clear that both considered models should behave in a similar manner near the dense region at $\mathbf{k} = \mathbf{0}$. Let us first consider the condensation dynamics of the SM. We write

$$M(k,t) = 2T\left[k - k_0 \arctan\left(\frac{k}{k_0}\right)\right] + F(k,k_0), \quad (52)$$

where $k_0(t)$ is uniquely defined by the condition

$$\rho(k=0,t) = \frac{2T}{k_0^2}, \quad (53)$$

ensuring that

$$F(k \to 0, k_0) \sim k^5. \quad (54)$$

The normalization condition reads

$$M(k=\Lambda,t) = 1 = 2T\left[\Lambda - k_0 \arctan\left(\frac{\Lambda}{k_0}\right)\right] + F(\Lambda,k_0). \quad (55)$$

We now look for a scaling solution of the form

$$F(k,k_0) = k_0^{\alpha} f\left(\frac{k}{k_0}\right), \quad (56)$$

and introduce the scaling variable

$$x = \frac{k}{k_0}. \quad (57)$$

This leads to

$$M'(k,t) = k^2\rho(k,t) = \frac{2Tk^2}{k^2 + k_0^2} + k_0^{\alpha-1} f'\left(\frac{k}{k_0}\right), \quad (58)$$

where, from now on, $M'(k,t)$ will denote the derivative with respect to momentum. In analogy with the equilibrium density Eq. (41), we observe that $\mu(t) = \beta k_0^2(t)/2$ can be seen as an effective chemical potential which is expected to vanish as time increases.

In $d=3$, the dynamical equation for $M(k,t)$ is

$$\frac{\partial M}{\partial t} = T\left(M'' - \frac{2}{k}M'\right) + \frac{M'^2}{k}. \quad (59)$$

Inserting the scaling ansatz, we find

$$\frac{\partial M}{\partial t} = k_0^{\alpha-2} T\left(f'' + 2\frac{x^2 - 1}{x(x^2+1)}f'\right) + k_0^{2\alpha-3}\frac{f'^2}{x}$$
$$= -2\dot{k}_0 T\left(\arctan(x) - \frac{x}{x^2+1}\right) + \dot{k}_0 k_0^{\alpha-1}(\alpha f - x f'). \quad (60)$$

It is clear that for $\alpha = 1$, all the terms of this equation scale the same way, if one chooses $\dot{k}_0 \sim k_0^{-1}$, i.e., $k_0(t) \sim (t_{coll} - t)^{1/2}$. However, we were able to prove analytically that all the solutions of the resulting scaling equation are nonphysical, leading to a density going to a constant for $k \gg k_0$. In addition, when we wrote Eq. (52), we implicitly assumed that the second term of the right-hand side (RHS) is a correction to the first one, which implies $\alpha > 1$. Hence we now consider $\alpha > 1$, and matching the leading terms of Eq. (60), we conclude that

$$\dot{k}_0 = -ck_0^{\alpha-2}, \tag{61}$$

and that the scaling function introduced above satisfies

$$f'' + 2\frac{x^2-1}{x(x^2+1)}f' = 2c\left(\arctan(x) - \frac{x}{x^2+1}\right), \tag{62}$$

which can be integrated once, leading to

$$f'(x) = \frac{2c}{3}\frac{x}{(x^2+1)^2}[(x^4+6x^2-3)\arctan(x) + 3x - 2x^3 - 4x\ln(x^2+1)]. \tag{63}$$

Note that $f'(x)/x^2$ is exactly the scaling function for the density as can be seen from Eq. (58). We give below some asymptotic results for $f(x)$ which will prove useful later:

$$f(x) = \frac{\pi c}{6}x^2 - \frac{4c}{3}x + 2\pi c\ln x + O(1), \quad x \to +\infty \tag{64}$$

$$= \frac{2c}{15}x^5 - \frac{8c}{45}x^7 + \frac{227c}{1260}x^9 + O(x^{11}), \quad x \to 0. \tag{65}$$

Let us now analyze the validity of the scaling regime. We ask that the neglected terms in Eq. (60) remain small in the range $k_0 \ll k \ll \Lambda$. Using the large x asymptotic for $f(x)$, we find that the nonlinear term in f appearing in Eq. (60) can be neglected, if

$$k_0^{2\alpha-3}\frac{f'^2}{x} \ll k_0^{\alpha-2} \quad \Rightarrow \quad k \ll k_0^{-(\alpha-2)}, \tag{66}$$

implying $\alpha \geq 2$. Similarly, the term arising from the time derivative of the residual density is negligible provided that

$$\dot{k}_0 k_0^{\alpha-1}(\alpha f - xf') \ll k_0^{\alpha-2} \Rightarrow k \ll k_0^{\frac{3-\alpha}{2}} \quad (\alpha \neq 2) \tag{67}$$

$$\Rightarrow k \ll \Lambda \quad (\alpha = 2), \tag{68}$$

which implies that $\alpha \geq 3$ or $\alpha=2$. We will find in the next sections that the admissible values $\alpha=3$ and $\alpha=2$ are in fact, respectively, associated to the collapse dynamics at T_c and strictly below T_c.

We would like to emphasize that the present model provides an interesting example where one obtains a scaling solution which scales differently from what would have been obtained from a naive power counting, assuming that *all* terms in Eq. (60) scale the same way (which would lead to $\alpha=1$).

B. Collapse at $T=T_c$

At T_c, the constant appearing in the scaling function is denoted c_c. Expressing the conservation of mass, we find

$$1 - 2T_c\left[\Lambda - k_0\arctan\left(\frac{\Lambda}{k_0}\right)\right] = F(\Lambda,k_0) \sim k_0^{\alpha}f\left(\frac{\Lambda}{k_0}\right), \tag{69}$$

$$\frac{\pi}{2}\Lambda^{-1}k_0 \sim \frac{\pi c_c}{6}\Lambda^2 k_0^{\alpha-2} + O(k_0^2), \tag{70}$$

which implies that

$$\alpha = 3, \quad c_c \sim \Lambda^{-3} \sim T_c^3. \tag{71}$$

Inserting this value for α in Eq. (61), we find that $k_0(t)$ decays exponentially at T_c, with a rate c_c. To complete the computation of the scaling function, we need to determine the constant c_c appearing as a prefactor in Eq. (63), and which also controls this exponential decay. Although the functional form of the scaling functions is identical for the SM and BEM, we will see that the constant c_c will differ for both models, as this constant is not entirely determined by the dynamics in the region of high density, where the term $(\rho+1)$ can be safely replaced by ρ. To make this point clearer, we now look for a global solution for the correction term $F(k,k_0)$, valid for momentum much greater than k_0, and up to $k=\Lambda$:

$$F(k,k_0) \approx k_0\phi(k), \quad k_0 \ll k \leq \Lambda. \tag{72}$$

Note that the large $x=\frac{k}{k_0}$ asymptotic of the scaling function should match the small k behavior of $\phi(k)$:

$$k_0\phi(k) \sim_{k\to 0} k_0^3 f\left(\frac{k}{k_0}\right) \sim_{k\gg k_0} \frac{\pi c_c}{6}k_0 k^2. \tag{73}$$

For the SM, expanding the integrated density up to leading order in k_0

$$M(k,t) = 2T_c\left(k - \frac{\pi}{2}k_0\right) + k_0\phi(k) + O(k_0^2), \tag{74}$$

we immediately find that

$$\phi(\Lambda) = \pi T_c. \tag{75}$$

On the other hand, substituting Eq. (74) into Eq. (40) we obtain to leading order in k_0:

$$\frac{\partial M}{\partial t} = k_0 T_c\left(\phi'' + \frac{2}{k}\phi'\right) = \dot{k}_0[\phi(k) - \pi T_c] = -c_c k_0[\phi(k) - \pi T_c], \tag{76}$$

so that $\phi(k)$ satisfies

$$\phi'' + \frac{2}{k}\phi' - \beta_c c_c[\phi(\Lambda) - \phi] = 0. \tag{77}$$

This equation admits the solution

$$\phi(k) = \pi T_c\left(1 - \frac{\sin(\sqrt{\beta_c c_c}k)}{\sqrt{\beta_c c_c}k}\right), \tag{78}$$

and expressing the condition of Eq. (75), we determine c_c and the full function $\phi(k)$

$$c_c = 4\pi^2 T_c^3, \quad \phi(k) = \pi T_c\left(1 - \frac{\sin(\pi k/\Lambda)}{\pi k/\Lambda}\right). \tag{79}$$

It is straightforward to check that $\phi(k)$ obeys the small k behavior of Eq. (73), identical to the $k \gg k_0$ behavior of the scaling function f.

Finally, the collapse at T_c for the SM has been fully analyzed, and the central density (53) diverges as

$$\rho(0,t) = 2T_c k_0^{-2}(t) \sim \exp(8\pi^2 T_c^3 t). \quad (80)$$

For infinite time, one converges to the expected equilibrium solution

$$M(k,\infty) = 2T_c k, \quad \rho(k,\infty) = 2T_c k^{-2}. \quad (81)$$

Now, this approach can be repeated for the BEM, starting with a similar ansatz

$$M(k,t) = \int_0^k \frac{k'^2}{\exp\left(\beta_c \frac{k'^2+k_0^2}{2}\right) - 1} dk' + k_0 \phi(k), \quad (82)$$

where $\mu(t) = \beta_c k_0^2/2$ acts like a time-dependent chemical potential. For $k \gg k_0$, we find that

$$\int_0^k \frac{k'^2}{\exp\left(\beta \frac{k'^2+k_0^2}{2}\right) - 1} dk' = \int_0^k \frac{k'^2}{\exp\left(\beta \frac{k'^2}{2}\right) - 1} dk' - \pi T k_0 + O(k_0^2). \quad (83)$$

Using the above result, we can insert the expression of $M(k,t)$ from Eq. (82) into the dynamical equation, Eq. (59), and find up to the leading order in k_0

$$\frac{\partial M}{\partial t} = k_0 T_c \left[\phi'' + \left(\frac{\beta_c k}{\tanh\left(\frac{\beta_c k^2}{4}\right)} - \frac{2}{k} \right) \phi' \right] = \dot{k}_0[\phi(k) - \pi T_c] = -c_c k_0 [\phi(k) - \pi T_c], \quad (84)$$

or, after simplification,

$$\phi'' + \left(\frac{\beta_c k}{\tanh\left(\frac{\beta_c k^2}{4}\right)} - \frac{2}{k} \right) \phi' - \beta_c c_c [\phi(\infty) - \phi] = 0, \quad (85)$$

with

$$\phi(\infty) = \pi T_c. \quad (86)$$

Now, c_c is selected by imposing that $\phi(k)$ is an increasing function (being the integral of a density) and that one gets a fast decay of $\phi(\infty) - \phi(k) \sim \exp(-\beta k^2/2)$. Solving this eigenvalue problem numerically, we find

$$c_c = 1.38452425\ldots = C\pi^2 T_c^3, \quad C = 1.50380614\ldots \quad (87)$$

which leads to

$$\rho(0,t) = 2T_c k_0^{-2}(t) \sim \exp(2C\pi^2 T_c^3 t). \quad (88)$$

In Fig. 1, we have plotted the function $\phi(k)$ for the BEM, compared to its analytical counterpart for the SM. As expected, they significantly differ only for large momenta. In Fig. 2, we numerically confirm the exponential growth of the density of particles at $\mathbf{k}=\mathbf{0}$ for both considered models, with the growth rate c_c in perfect agreement with the present theoretical analysis. Finally, in Fig. 3, we illustrate the scaling behavior of the density profile.

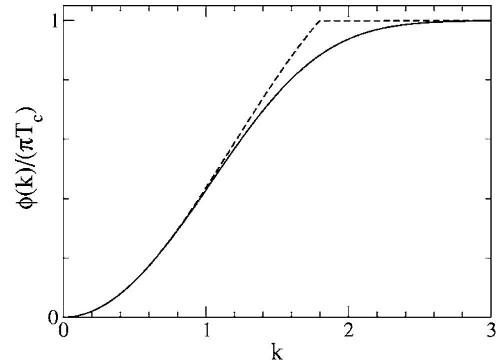

FIG. 1. We plot the analytical expressions of $\phi(k)$ [see Eq. (79), dashed line] for the simplified model compared to the numerical solution of Eq. (85) (full line). We have chosen the size of the confining box to be $\Lambda = 1.79805\ldots$ so that the small k behavior of the two functions coincides.

C. Collapse for $0 < T < T_c$

We now repeat the above procedure below T_c. For the SM model, expressing conservation of mass, we obtain

$$1 - 2T\left[\Lambda - k_0 \arctan\left(\frac{\Lambda}{k_0}\right)\right] = F(\Lambda, k_0) \sim k_0^\alpha f\left(\frac{\Lambda}{k_0}\right), \quad (89)$$

$$\frac{T_c - T}{T_c} \sim \frac{\pi c}{6} \Lambda^2 k_0^{\alpha-2} + O(k_0), \quad (90)$$

which implies that

$$\alpha = 2, \quad c(T) \sim T_c(T_c - T). \quad (91)$$

For this value of α, Eq. (61) can be integrated, leading to

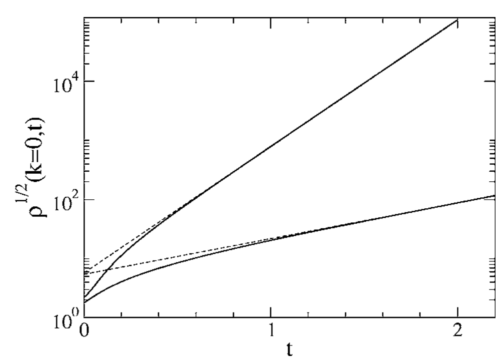

FIG. 2. At $T=T_c$, we plot $\rho^{1/2}(k=0,t) \sim k_0^{-1}(t) \sim \exp(c_c t)$ computed numerically by integrating the dynamical equation for $M(k,t)$ [see Eq. (59)], for the SM ($\Lambda=1$, top full line) and the BEM (bottom full line). In both cases, we find an exponential growth, with a rate in perfect agreement with our exact results for c_c (the straight dashed lines in these semilog plots have a slope equal to the theoretical values for c_c). For both models, we obtain an excellent fit of $\rho^{1/2}(k=0,t)$ to the functional form $\rho^{1/2}(k=0,t) = A\exp(c_c t) - B$, with $A \sim 5.7$ and $B \sim 0.5$ (fit not shown, but indistinguishable from data starting from $t \sim 0.2$).

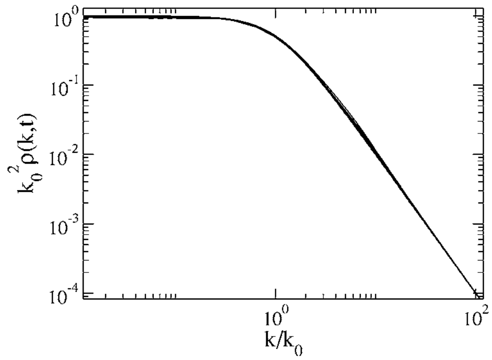

FIG. 3. At $T=T_c$, we plot $(2T_c)^{-1}k_0^2\rho(k,t)$ as a function of $x=k/k_0$ for the BEM, for times for which the central density $\rho(0,t)=2T_c k_0^{-2}$ is equal to 100×2^n $(n=1,\ldots,14)$. We have also plotted the corresponding scaling function $1/(1+x^2)$ (dashed line), which is perfectly superimposed with the data collapse. A similar plot for the SM would be indistinguishable, for the scale of momentum shown. Below T_c, a similar scaling arises for both models.

$$k_0(t) = c(T)(t - t_{coll}). \tag{92}$$

We thus find that, below T_c, a singularity should arise in a finite time, for which the central density diverges when t goes to t_{coll}.

In order to obtain analytical results, we focus on the collapse dynamics just below T_c, so that $\frac{T-T_c}{T_c}\ll 1$, hence $F(k,k_0)\ll 1$ and $c\ll 1$. In this regime, nonlinear terms in F can be safely neglected. For momenta $k\gg k_0$, we define a function $\phi(k)$ such that

$$F(k,k_0) \approx \left(1+\frac{\lambda}{c}k_0\right)\phi(k), \quad k_0 \ll k \le \Lambda, \tag{93}$$

resulting in the following expression for the integrated mass density:

$$M(k,t) = 2T\left(k-\frac{\pi}{2}k_0\right) + \left(1+\frac{\lambda}{c}k_0\right)\phi(k) + O(k_0^2). \tag{94}$$

Taking $k=\Lambda$ and matching terms of order $O(k_0^0)$ and $O(k_0^1)$, we deduce that

$$\phi(\Lambda) = \frac{T_c - T}{T_c} = \pi\frac{c}{\beta\lambda}. \tag{95}$$

Inserting the expression of $M(k,t)$ in the dynamical equation Eq. (59), we obtain

$$\frac{\partial M}{\partial t} = T\left(\phi'' + \frac{2}{k}\phi'\right) = \pi c T - \lambda\phi, \tag{96}$$

or

$$\phi'' + \frac{2}{k}\phi' - \beta\lambda[\phi(\Lambda) - \phi] = 0. \tag{97}$$

We recognize the very same equation for $\phi(k)$ as in the preceding section, so that the two functions are identical up to a multiplicative constant. Hence, for the SM, we obtain

$$\lambda = c_c = 4\pi^2 T_c^3, \quad c(T) = 4\pi T_c(T_c - T), \tag{98}$$

$$\phi(k) = \frac{T_c - T}{T_c}\left(1 - \frac{\sin(\pi k/\Lambda)}{\pi k/\Lambda}\right),$$

and a power law divergence of the central density

$$\rho(0,t) = 2Tk_0^{-2}(t) \sim \frac{1}{8\pi^2 T_c^3}\left(\frac{T_c - T}{T_c}\right)^{-2}(t_{coll} - t)^{-2}. \tag{99}$$

For the SM, we can evaluate $\phi(k)$, $c(T)$, and $\lambda(T)$ at the next order in $\delta=\frac{T_c-T}{T_c}$. We write

$$\phi(k) = \delta\phi_0(k) + \delta^2\phi_1(k), \tag{100}$$

$$c(T) = \delta c_0 + \delta^2 c_1, \tag{101}$$

$$\lambda(T) = \lambda_0 + \delta\lambda_1, \tag{102}$$

where $\phi_0(k)$, c_0, and $\lambda_0=c_c$ can be easily determined from Eq. (98). Inserting the above ansatz in the dynamical equation leads to a complicated linear equation for $\phi_1(k)$. We must impose the boundary conditions $\phi_1(0)=\phi_1'(0)=0$ and $\phi_1(\Lambda)=0$, which fix the value of c_1 and λ_1. After some cumbersome but straightforward calculations, we obtain

$$\begin{aligned}\phi_1\left(x=\pi\frac{k}{\Lambda}\right) = \frac{\sin x}{x}\Bigg[&\frac{c_1}{8\pi T_c^2}\left(1+\frac{\pi - x}{\tan(x)}\right) + \frac{2\pi\cos(x)}{3x} \\ &+ \frac{3\pi}{4\tan(x)}\left(\int_\pi^x \frac{\cos(t)}{t}dt - \int_{3\pi}^{3x}\frac{\cos(t)}{t}dt\right) \\ &+ \frac{\pi}{4}\left(3\int_{3x}^{+\infty}\frac{\sin(t)}{t}dt - \frac{11}{3}\int_x^{+\infty}\frac{\sin(t)}{t}dt\right) \\ &+ \frac{\pi[\cos(2x)-1]}{6x^2\sin(x)}\Bigg],\end{aligned} \tag{103}$$

with

$$c_1 = \frac{2\pi T_c^2}{3}\left(9\ln(3) - 4 - 9\int_\pi^{3\pi}\frac{\cos(t)}{t}dt\right) \tag{104}$$

$$= C_1 T_c^2, \quad C_1 = 13.51919467982\ldots \tag{105}$$

and

$$\lambda_1 = \pi C_1 T_c^3. \tag{106}$$

Finally, we obtain

$$c(T) = 4\pi T_c(T_c - T) + C_1(T_c - T)^2 + O((T - T_c)^3), \tag{107}$$

$$\lambda(T) = 4\pi T_c^3 + \pi C_1 T_c^2(T_c - T) + O(T_c(T - T_c)^2), \tag{108}$$

and note that the next correction to $c(T)$ and $\lambda(T)$ are both positive.

Again, this approach can be repeated for the more realistic BEM, by writing

$$M(k,t) = \int_0^k \frac{k'^2}{\exp\left(\beta\frac{k'^2+k_0^2}{2}\right) - 1} dk' + \left(1 + \frac{\lambda}{c}k_0\right)\phi(k). \tag{109}$$

At the leading order in k_0, the dynamical equation, Eq. (59), leads to

$$\frac{\partial M}{\partial t} = T\left[\phi'' + \left(\frac{\beta k}{\tanh\left(\frac{\beta k^2}{4}\right)} - \frac{2}{k}\right)\phi'\right] = \pi c T - \lambda\phi, \tag{110}$$

or more simply to

$$\phi'' + \left(\frac{\beta k}{\tanh\left(\frac{\beta k^2}{4}\right)} - \frac{2}{k}\right)\phi' - \beta\lambda[\phi(\infty) - \phi] = 0, \tag{111}$$

which is again the same equation as the one found in the previous section, which determines the function $\phi(k)$ up to a multiplicative constant. In order to compute this constant, we have to match the terms of order $O(k_0^0)$ and $O(k_0^1)$ for $k \to +\infty$ in Eq. (109), which implies that

$$\phi(\infty) = 1 - \left(\frac{\beta_c}{\beta}\right)^{3/2} \approx \frac{3}{2}\frac{T_c - T}{T_c} \tag{112}$$

$$= \pi\frac{c}{\beta\lambda}. \tag{113}$$

Finally, we obtain

$$\lambda = c_c = C\pi^2 T_c^3, \quad c(T) = C'\pi T_c(T_c - T), \tag{114}$$

$$C' = \frac{3C}{2} = 2.2557092066\ldots$$

and

$$\rho(0,t) = 2Tk_0^{-2}(t) \sim \frac{2}{C'^2\pi^2 T_c^3}\left(\frac{T_c - T}{T_c}\right)^{-2}(t_{coll} - t)^{-2}. \tag{115}$$

Note that at $t = t_{coll}$, we find that the density and the integrated density are

$$\rho(k,t_{coll}) = \frac{2T}{k^2} + \frac{\phi'(k)}{k^2}, \quad M(k,t_{coll}) = 2Tk + \phi(k) \tag{116}$$

for the simplified model, and

$$\rho(k,t_{coll}) = \frac{1}{\exp\left(\beta\frac{k^2}{2}\right) - 1} + \frac{\phi'(k)}{k^2}, \tag{117}$$

$$M(k,t_{coll}) = \int_0^k \frac{k'^2}{\exp\left(\beta\frac{k'^2}{2}\right) - 1} dk' + \phi(k)$$

for the more realistic Bose-Einstein model. In addition to the equilibrium density, we obtain in both models a singular contribution near $k=0$

$$\frac{\phi'(k)}{k^2} \sim \frac{\pi}{3k}c(T), \quad k \to 0. \tag{118}$$

However, we do not observe the appearance of a Dirac peak at $\mathbf{k}=\mathbf{0}$, implying that the evolution of the system is not finished yet. Hence, we expect that after t_{coll}, the mass included in $\phi(k)$ will be swallowed at $k=0$, finally giving rise to the condensate.

Before addressing this issue, we give an estimate of t_{coll}, near T_c. If one is very close to T_c, the density first grows exponentially like at T_c, before crossing over to the behavior of Eq. (115), when the time is of order t_{coll}. Matching the two regimes, we obtain

$$\rho(k=0,t \sim t_{coll}) \sim \exp(2c_c t_{coll}) \sim \left(\frac{T_c - T}{T_c}\right)^{-2} t_{coll}^{-2}, \tag{119}$$

leading to the rough estimate

$$t_{coll} \sim T_c^{-3} \ln\left(\frac{T_c}{T_c - T}\right). \tag{120}$$

As expected, t_{coll} diverges as the temperature approaches T_c. This analysis suggests a global form for $\dot{k}_0(t)$, valid for all temperature

$$\dot{k}_0 = -c(T)h\left(\frac{c_c}{c(T)}k_0\right), \quad h(0) = 1, \quad h(x) \sim_\infty x, \tag{121}$$

where $c(T) \sim T_c(T_c - T)$ for both models. If we assume the simple form $h(x) = 1 + x$ [numerically we find $h(x) = 1 + Ax$, for small x; see Fig. 5], which is compatible with the known asymptotics of $f(x)$ in Eq. (121), we obtain

$$\dot{k}_0 = -c_c k_0 - c(T). \tag{122}$$

This equation has a global solution

$$k_0(t) = \frac{c(T)}{c_c}\{\exp[c_c(t_{coll} - t)] - 1\}. \tag{123}$$

For times t close to t_{coll}, this expression leads to the asymptotic scaling of Eq. (92), whereas it also reproduces the early exponential decay with rate c_c, which is expected for T close to T_c. Imposing that at $t=0$, k_0 should be of order of a typical momentum of the initial condition (for instance $k_0 \sim \Lambda$ for the SM), we recover exactly the estimate of t_{coll} obtained in Eq. (120).

In Fig. 4, we plot the numerical estimate of $\phi(k)$ obtained by numerically integrating Eq. (59) at $T = 0.9T_c$, and compare it to our theoretical results, which are, in principle, only valid very close to T_c. In Fig. 5, we plot $\dot{k}_0(t) \to -c(T)$ obtained by numerically integrating Eq. (59) and find a fair agreement

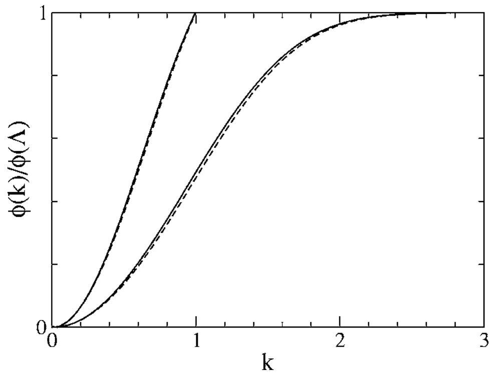

FIG. 4. For the simplified model, we plot $\phi(k)=M(k,t_{coll})-2Tk$ obtained by numerically integrating the dynamical equation for $M(k,t)$ at $T=0.9T_c$ (top full line). We have chosen $\Lambda=1$ so that $T_c=1/2$ and $\phi(\Lambda)=\frac{T_c-T}{T_c}=1/10$ [see Eq. (95)]. It is in good agreement with our analytical expression of Eq. (98) (top dashed line), which is strictly valid only very close to T_c. We also plot the numerical (bottom full line) and theoretical expression of $\phi(k)$ (bottom dashed line), for the Bose-Einstein model $[\phi(\Lambda=\infty)=1-27/10^{3/2}=0.1462\ldots\approx\frac{3}{2}\frac{T_c-T}{T_c}]$.

with our theoretical estimates for $c(T)$, which are strictly valid only close to T_c.

D. Collapse at $T=0$

For completeness, we now consider the collapse dynamics at $T=0$. Since we are interested in the density scaling function, we address this question within the simplified model. We consider the dynamical equation for the density rather than for the integrated density

$$\frac{\partial\rho}{\partial t}=3\rho^2+2k\rho\rho'. \tag{124}$$

Inserting a scaling ansatz of the form

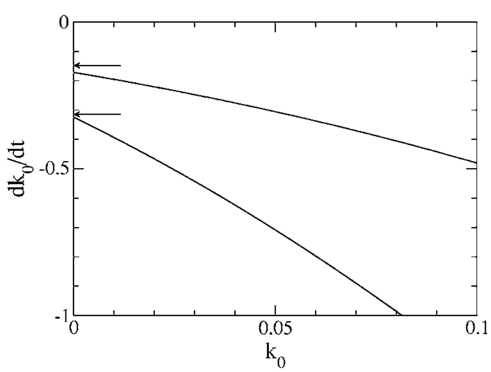

FIG. 5. For the simplified model ($\Lambda=1$, bottom line) and the Bose-Einstein model (top line), we plot $\dot{k}_0(t)\to -c(T)$ as a function of $k_0(t)$ as obtained by integrating the dynamical equation for $M(k,t)$ at $T=0.9T_c$. The theoretical slopes $-c(T)$ evaluated at the first order in T_c-T are indicated by an arrow. We find that the numerical value of $c(T)$ is slightly bigger than our first order calculation, consistent with the fact that the next correction is analytically found to be positive for the SM [see Eq. (107)]. For small $k_0(t)$ (i.e., close to t_{coll}), we find $\dot{k}_0(t)\approx -c(T)-Ac_ck_0(t)$, with $A\approx 1.35$ for the SM, and $A\approx 1.65$ for the BEM.

$$\rho(k,t)=\rho_0 g\left(\frac{k}{k_0}\right), \tag{125}$$

in the dynamical equation Eq. (124), we obtain

$$\dot{\rho}_0 g-\rho_0\frac{\dot{k}_0}{k_0}xg'=\rho_0^2(3g^2+2xgg'). \tag{126}$$

We introduce the parameter α defined by

$$\frac{\dot{\rho}_0}{\rho_0}=-\alpha\frac{\dot{k}_0}{k_0}=\alpha\rho_0. \tag{127}$$

Then, the equation for the scaling function becomes

$$\frac{1-2g}{g(\alpha-3g)}g'=-\frac{1}{x}, \tag{128}$$

which can be exactly solved, leading to the following implicit equation for $g(x)$:

$$g(x)\left(\frac{\alpha}{3}-g(x)\right)^{2\alpha/3-1}=Cx^{-\alpha}, \tag{129}$$

where C is a constant depending on the initial conditions, as was already noticed within the study of the gravitational collapse at $T=0$ [29–31]. Now, for small $x=k/k_0$, $g(x)$ must be an analytic function, with a small x expansion of the form $g(x)=g(0)+g''(0)x^2/2+\cdots$. Matching the small x behavior of the right-hand side and left-hand side of Eq. (129), we obtain

$$\alpha=\frac{6}{7},\quad g(0)=\frac{2}{7}, \tag{130}$$

and

$$g(x)\left(\frac{2}{7}-g(x)\right)^{-3/7}=Cx^{-6/7}. \tag{131}$$

We also obtain the exact expression for the central density, and the time-dependent width of the dense core $k_0(t)$

$$\rho(k=0,t)=\frac{2}{7}\rho_0(t)=\frac{1}{3}(t_{coll}-t)^{-1},\quad k_0(t)=(t_{coll}-t)^{7/6}. \tag{132}$$

Finally, at $t=t_{coll}$, the small k behavior of the density becomes universal

$$M(k,t_{coll})\sim k^{15/7},\quad \rho(k,t_{coll})\sim k^{-6/7}, \tag{133}$$

in contrast with the behavior obtained for $0<T\leq T_c$, where we found $\rho(k,t_{coll})\sim 2Tk^{-2}$.

IV. POSTCOLLAPSE DYNAMICS FOR $0<T<T_c$

As mentioned in Sec. III C, the density profile at $t=t_{coll}$ is not yet equal to the equilibrium profile which presents a Dirac peak at $\mathbf{k}=\mathbf{0}$. This means that after t_{coll}, the density profile should continue to relax to

$$\rho(\mathbf{k})=\frac{T_c-T}{T_c}\delta(\mathbf{k})+2Tk^{-2} \tag{134}$$

for the simplified model, and

$$\rho(\mathbf{k}) = \left[1 - \left(\frac{\beta_c}{\beta}\right)^{3/2}\right]\delta(\mathbf{k}) + \frac{1}{\exp\left(\beta\frac{k^2}{2}\right) - 1} \tag{135}$$

for the Bose-Einstein model. In order to obtain analytical results in this postcollapse stage, we again consider the case of a temperature just below T_c, such that

$$\frac{T_c - T}{T_c} \ll 1. \tag{136}$$

In this regime, and for the SM, the following ansatz is an exact solution of the dynamical equation, Eq. (59)

$$M(k,t) = M_0(t) + 2Tk + \left(1 - \frac{M_0(t)}{\phi(\Lambda)}\right)\phi(k), \tag{137}$$

where $M_0(t)$ is the time-dependent mass in the condensate, and where $\phi(k)$ has been calculated in the collapse regime in the previous section. Inserting this ansatz in the kinetic equation, we obtain

$$\frac{\partial M}{\partial t} = \dot{M}_0\left(1 - \frac{\phi(k)}{\phi(\Lambda)}\right) = T\left(1 - \frac{M_0(t)}{\phi(\Lambda)}\right)\left(\phi'' + \frac{2}{k}\phi'\right), \tag{138}$$

where we have neglected quadratic terms in $\phi(k)$, which is justified near T_c. For this equation to be compatible with Eq. (97), i.e., the defining equation for $\phi(k)$, we must have

$$\dot{M}_0 = c_c[\phi(\Lambda) - M_0(t)], \tag{139}$$

which can be easily solved, leading to the full time dependence of the condensate mass $M_0(t)$

$$M_0(t) = \frac{T_c - T}{T_c}(1 - \exp[-4\pi^2 T_c^3(t - t_{coll})]). \tag{140}$$

For the Bose-Einstein model, we proceed in a similar manner, and find for T close to T_c

$$M_0(t) = \left[1 - \left(\frac{\beta_c}{\beta}\right)^{3/2}\right](1 - \exp[-C\pi^2 T_c^3(t - t_{coll})]) \tag{141}$$

$$\approx \frac{3}{2}\frac{T_c - T}{T_c}(1 - \exp[-C\pi^2 T_c^3(t - t_{coll})]) \tag{142}$$

For both models, the condensate mass initially grows linearly with time

$$M_0(t) = \pi T_c c(T)(t - t_{coll}) \sim T_c^2(T_c - T)(t - t_{coll}), \tag{143}$$

$$t - t_{coll} \ll T_c^{-3},$$

whereas it saturates exponentially fast to its equilibrium value, with a rate c_c equal to the relaxation rate found at T_c (see Sec. III B and the next section), since one can write

$$1 - \frac{M_0(t)}{M_0(\infty)} = \exp[-c_c(t - t_{coll})]. \tag{144}$$

V. RELAXATION TIME FOR $T > T_c$

Finally, in this section, we address the problem of determining the relaxation rate to the equilibrium solution above T_c. Writing

$$M(k,t) = \int_0^k \frac{k'^2}{\exp\left(\frac{\beta k'^2}{2} + \mu\right) - 1}dk' + \epsilon(k)\exp(-t/\tau), \tag{145}$$

and neglecting nonlinear terms in $\epsilon(k)$, we find that

$$\epsilon(k) = h(\sqrt{\beta}k), \tag{146}$$

where $h(x)$ satisfies

$$h'' + \left(\frac{x}{\tanh\left(\frac{x^2}{4} + \frac{\mu}{2}\right)} - \frac{2}{x}\right)h' + \tau^{-1}h = 0. \tag{147}$$

For $T \gg T_c$ (and hence $\mu \gg 1$), we can replace tanh by unity and the solution can be exactly written in terms of the Kummer confluent hypergeometric function K

$$h(x) = x^3 K\left(\frac{3\tau + 1}{2\tau}, \frac{5}{2}, -\frac{x^2}{2}\right). \tag{148}$$

Imposing a fast decaying solution, we find

$$h(x) = x^3 \exp\left(-\frac{x^2}{2}\right), \quad \tau(T \to +\infty) = \frac{1}{2}. \tag{149}$$

For the simplified model, the limit of high temperature is not physically interesting since, due to the finite box, the particles cannot spread up to large momenta.

For $0 < T - T_c \ll T_c$ (and hence $\mu \ll 1$), the eigenequation, Eq. (147), coincides with the defining equation for the function $\phi(k)$ introduced in Sec. III B. Hence $\tau(T \to T_c) = c_c^{-1}$, using the respective values of c_c for the two models considered.

It seems paradoxical that the relaxation time does not diverge at T_c, contrary to what is expected for a continuous phase transition. However, τ is *not* the equilibration time, which is the typical time τ_{eq} needed to reach the equilibrium solution. This time can be evaluated by considering that close but above T_c, the chemical potential initially decreases as $\beta k_0(t)^2/2$ does at T_c, and reaches the order of magnitude of its equilibrium value after a time τ_{eq}. Thus we write

$$k_0(\tau_{eq}, T_c) \sim \exp(-c_c\tau_{eq}) \sim k_0(T) = \sqrt{2\mu T}. \tag{150}$$

Using the expressions for μ near T_c given in Eq. (36) for the BEM and in Eq. (47) for the SM, we obtain

$$\tau_{eq} \sim -T_c^{-3}\ln\mu \sim T_c^{-3}\ln\left(\frac{T_c}{T - T_c}\right) \tag{151}$$

for both models. Note the similarity of this estimate with the expression of t_{coll} near T_c, given in Eq. (120). The exponential relaxation controlled by τ only starts after a time of order $\tau_{eq} \gg \tau$.

VI. COMPARISON WITH OTHER WORKS

In this section, we give a short review of classical works concerning the dynamics of the Bose-Einstein condensation to show how our results compare with these studies. The dynamics of the Bose-Einstein condensation has been described by two apparently different kinetic theories. The first kinetic theory is based on a quantum version of the Boltzmann equation for the one-particle distribution function $\rho(\mathbf{k},t)$. This is a semiclassical approach which introduces corrections for quantum statistics into the ordinary Boltzmann collision term [10–12]. Another description is provided by the time-dependent Gross-Pitaevskii equation for the condensate wave function $\psi(\mathbf{x},t)$ (order parameter for the Bose condensate) in a spatially homogeneous medium [39]. This equation does not take into account quantum fluctuations, or thermal or irreversible effects, but is valid when a large number of particles have condensed.

One of the first solutions of these kinetic equations is due to Levich and Yakhot [40] who consider a gas of bosons without interaction (i.e., neglecting collisions) in contact with a thermal bath of fermions. They provide an analytical solution of the corresponding quantum Boltzmann equation and find that the Bose condensate forms in an infinite time. Since this result is inconsistent with observations, Stoof [41] argues that the quantum Boltzmann equation is not valid to describe the condensate. He considers an isolated gas of bosons in interaction (i.e., with interatomic collisions) and argues that the dynamics of the collapse follows three steps: (i) an incoherent evolution described by the quantum Boltzmann equation, (ii) a coherent evolution triggering a phase transition (instability) leading to the Bose condensate, and (iii) a thermalization between the condensate and the noncondensed atoms interpreted as quasiparticles in the sense of Bogoliubov. Stoof focuses on the coherent evolution. Using a functional formulation of the Keldysh theory, he derives a time dependent Landau-Ginzburg equation for the order parameter $\langle\psi(\mathbf{x},t)\rangle$ of the phase transition. He also shows that the transition temperature for interacting bosons is larger than for an ideal gas and that the nucleation of the condensation is short, contrary to the result of [40], except for temperatures close to the critical temperature T_c. In a more recent paper, Stoof [42] derives a Fokker-Planck equation for the probability distribution of the order parameter $\langle\psi(\mathbf{x},t)\rangle$ which gives a unified description of both the incoherent (kinetic Boltzmann) and coherent (Gross-Pitaevskii) stages of Bose-Einstein condensation. A similar program of unification of the two theories is carried out by Gardiner and Zoller [43]. Using a projection of the density operator ρ, they derive a quantum kinetic master equation (QKME) for bosonic atoms and recover, in particular limits, the quantum Boltzmann equation and the Gross-Pitaevskii equation.

Semikoz and Tkachev [13] and Lacaze *et al.* [14] numerically solve the quantum Boltzmann equation and find the formation of a condensate in finite time and the growth of this condensate in a postcollapse regime. This is quite different from the results of Levich and Yakhot [40] which are valid when the system is coupled to a bath of fermions. Therefore the original bosonic Boltzmann kinetic equation (without fermion bath approximation) can account for a finite time formation of the condensate. More recently, using an analogy with optical turbulence, Connaughton and Pomeau [44] obtain a kinetic equation for the spectral particle density $\rho(\mathbf{k},t)$ directly from the Gross-Pitaevskii equation for the wave function $\psi(\mathbf{x},t)$ when the nonlinear term is considered as a perturbation ($\mathbf{k}$ is the conjugate of $\mathbf{x}$ in the Fourier analysis). They show that this kinetic equation has the same form as the quantum Boltzmann equation.

Several authors [13,14,45,46] have investigated self-similar solutions of the quantum Boltzmann equation in the form $\rho(\epsilon,t)=\rho_0(t)f[\epsilon/\epsilon_0(t)]$, where $\epsilon=\frac{k^2}{2m}$. In particular, Semikoz and Tkachev [13] find a precollapse regime generating, in a finite time t_{coll}, a distribution function of energies scaling as $f(\epsilon)\sim\epsilon^{-\alpha}$ with $\alpha=1.24$. This profile is slightly steeper than the Zakharov profile $\epsilon^{-7/6}$ providing an exact static solution of the quantum Boltzmann equation corresponding to a constant mass flux J in momentum space toward the condensate. The central density increases as $\rho_0(t)\sim(t_{coll}-t)^{-\alpha/2(\alpha-1)}\sim(t_{coll}-t)^{-2.6}$ and becomes infinite at $t=t_{coll}$. The typical core energy $\epsilon_0(t)\sim(t_{coll}-t)^{1/2(\alpha-1)}\sim(t_{coll}-t)^{2.1}$ goes to zero at $t=t_{coll}$. Semikoz and Tkachev [13] also investigate the postcollapse regime. They find that the energy distribution passes from $\epsilon^{-\alpha}$ to ϵ^{-1} after the singularity and that the mass of the condensate grows like $n_0(t)=(t-t_{coll})^{(3-2\alpha)/4(\alpha-1)}\sim(t-t_{coll})^{0.54}$ just after collapse. Lacaze *et al.* [14] also investigate self-similar solutions and obtain similar results with an exponent $\alpha=1.234$.

Our approach is physically different since we consider a gas of noninteracting bosons in contact with a heat bath establishing a Bose-Einstein distribution (canonical description) while the previous authors considered an isolated system of bosons in interaction (microcanonical description). However, although the dynamics differ in the details (as expected), the phenomenology of the collapse is almost the same and the simplified form of our kinetic equation (18) allows for a complete analytical solution of the problem (pre- and post-collapse) which is not possible for the quantum Boltzmann equation [13,14,45,46]. This is clearly an interest of our model.

VII. CONCLUSION

In the present paper, we have considered the condensation in $\mathbf{k}$-space of a homogeneous gas of noninteracting bosons in a thermal bath fixing the temperature T. In statistical mechanics, this corresponds to a *canonical* description. We have pointed out the striking analogy between the dynamical equation describing the Bose-Einstein condensation in the canonical ensemble and the one describing the evolution of a self-gravitating gas of Brownian particles (or the chemotaxis of bacterial populations).

In analogy to the case of a classical gas, we have constructed the dynamical Fokker-Planck equation for the momentum distribution. For $T<T_c$, the system experiences a finite time singularity, i.e., the density at $\mathbf{k}=\mathbf{0}$ becomes infinite in a finite time $t_{coll}\sim T_c^{-3}\ln\left(\frac{T_c}{T-T_c}\right)$. However, the "singularity contains no mass" and the condensate is formed during

the postcollapse regime for $t > t_{coll}$. This dynamical scenario leads, for $t \to +\infty$, to a Dirac peak in addition to the Bose-Einstein distribution with zero chemical potential, which is entirely consistent with the distribution predicted by Einstein at equilibrium [9]. Our canonical description permits a complete analytical treatment for the different stages of the process, contrary to the microcanonical approach starting from the semiclassical Boltzmann equation [13,14], where one has to mostly rely on numerical simulations. Although qualitatively similar (existence of a finite time precollapse regime for $t < t_{coll}$, scaling behavior of the density, formation of the condensate in the postcollapse regime,…), we note that our results differ quantitatively from the ones obtained within the microcanonical approach. For instance, $k_0(t) = c(T)(t_{coll} - t)$ in our canonical model, whereas $k_0(t) \sim (t_{coll} - t)^{\gamma}$ in the microcanonical ensemble, where, numerically, one finds $\gamma \approx 1.07$ (the difference is even more apparent on other quantities) [14]. This should not be surprising, as the same phenomenon arises in the study of the collapse dynamics of a self-gravitating gas, which is different in the canonical and microcanonical ensembles, though qualitatively similar in many respects [29].

In this paper, we have assumed that the Bose gas is homogeneous. The inhomogeneous situation could be studied semiclassically by introducing a mean-field advective term in the kinetic equation such that

$$\frac{\partial \rho}{\partial t} + \mathbf{k}\frac{\partial \rho}{\partial \mathbf{r}} - \nabla U(\mathbf{r})\frac{\partial \rho}{\partial \mathbf{k}} = Q(\rho), \tag{152}$$

where $Q(\rho)$ is either the bosonic Fokker-Planck operator of Eq. (18) in the presence of a thermal bath, or the bosonic Boltzmann operator when elastic collisions are taken into account. An even more general model could take into account both contributions, or include additional semiclassical contributions arising from the fact that the operators $\mathbf{r}$ and $\mathbf{k}$ do not commute.

As a final comment, we may note that there exists analogies between the Bose-Einstein condensation and the inverse cascade process in two-dimensional turbulence where energy accumulates at large scales ($k=0$) to form a macrovortex [47].

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